Green's function; $g_{ij}(x,y;\ \xi,\eta)$, elements of the Green's matrix; $\mu(x,y)$, density; λ_i, thermal conductivity coefficients.

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DETERMINATION OF THE GEOMETRIC-OPTICS COEFFICIENTS OF THERMAL RADIATION BY THE MONTE CARLO METHOD

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Algorithms of the Monte Carlo method to determine the governing angular coefficients for different formulations of the radiant exchange problem under conditions of a diathermal medium and results of their verification by means of exact solutions are presented.

The method of statistical tests, or the Monte Carlo method [1-6], has recently been applied quite frequently to the solution of applied radiant heat-transfer problems. In the case of systems filled with a diathermal medium, this method is used principally for the direct determination of the geometric-optics characteristics of the radiation field [2-5]. Let us examine the question of applying the Monte Carlo method to obtain directly one such characteristic, the governing angular coefficient [7, 8]. Let us take the usual assumptions about the diffuseness of the radiation and the grayness and opacity of the system boundaries. Let us limit ourselves to finding the mean value of the coefficient of greatest interest in engineering practice. Let us assume that the system under investigation consists of a finite number of zones (bodies), within each of whose limits the given optical and energetic characteristics are constant from point to point.

The possibility of a statistical modeling of the governing angular coefficient is based on its representation as an infinite functional series [7] (whose first member is the geometric angular coefficient, and the next terms of the series take into account the first, second, and all the remaining reflections) which expresses the method of multiple reflections explicitly. Therefore, the desired coefficient can be determined by observing the fate of the different rays in time. The probabilistic treatment of the mean governing angular coefficient Φ_{ik} as a characteristic of the fraction of proper radiation of the zone i reaching the zone k directly and taking into account all the re-reflections in the system is also used in constructing the algorithm of the Monte Carlo method.

The field of governing angular coefficients is ordinarily found from the solution of integral equations of the resolvent of the initial integral equations of radiation transfer.

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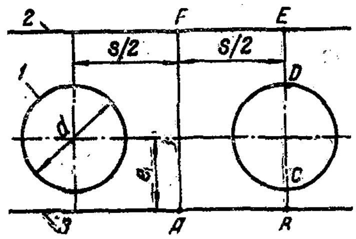

Fig. 1. Tubular system.

The Monte Carlo method, which permits the determination of these coefficients, is substantially an approximate method of solving the equations of the resolvent. In particular, in the case of diffuse reflection on the boundaries, the system of integral equations for the mean resolving angular coefficients becomes [7, 9]

$$\Phi_{ik}-\frac{1}{F_i}\sum_{j=1}^{n}R_j\int_{F_j}\varphi(\mathscr{P}_j,\ F_i)\,\Phi(\mathscr{P}_j,\ F_k)\,dF_{\mathscr{P}_j}=\varphi_{ik} \qquad (1)$$

$$(i,\ k=1,\ 2,\ \ldots,\ n).$$

The solution obtained by the Monte Carlo method has the exact solution of (1) as limit if the hypotheses used in deriving the integral equations and in the Monte Carlo method are equivalent. For example, the very same geometric configuration can be described either by a field of local geometric angular coefficients [in the integral equations (1)] or by analytic equations of the system surface (in the Monte Carlo method). Let us note that the latter description is considerably simpler.

In practice, the system of integral equations (1) is approximated in the majority of cases by a governing system of linear algebraic equations [7]:

$$\Phi_{ik}-\sum_{j=1}^{n}R_j\varphi_{ij}\Phi_{jk}=\varphi_{ik}\quad (i,\ k=1,\ 2,\ \ldots,\ n). \qquad (2)$$

The approximate solutions obtained from (2) can contain a noticeable error, whose determination is fraught with considerable difficulty [9, 10]. The complexity in estimating the error is inherent also in the majority of other numerical methods to solve the integral equations.

The accuracy of the angular coefficients found by the Monte Carlo method is usually estimated on the basis of parameters of the normal probability distribution law [1, 2] by starting from the Moivre–Laplace theorem [11]. This estimate shows that, for example, the number of histories should be about 25,000 for a confidence not below 0.9 in the result, and 50,000 for a confidence not below 0.97, to obtain an absolute error in the geometric angular coefficient of 0.005 which is acceptable in technical problems. It can be expected that the error in the quantity $\varepsilon_k\Phi_{ik}$ in determining the governing angular coefficient will correspond to the mentioned regularity, including the numerical example presented.

The fundamentals of constructing algorithms of the Monte Carlo method to find directly the governing angular coefficients for different formulations of the radiant heat-exchange problem in systems filled with a diathermal medium are elucidated below and the verification of these algorithms, in the form of programs for the "Minsk-2" electronic digital computer, is carried out by means of exact particular solutions.

1. The Fundamental Formulation of the Problem for a Diffuse Law of Radiant Flux Reflection by the System Boundaries

This formulation is characterized by giving the temperature field on the boundaries, and the integral equation (1) corresponds to it completely.

A "packet" of particles of unit "weight" is emitted from a random radiating point of the chosen body in a random direction drawn in conformity with Lambert's law. Upon meeting the system boundaries, a fraction of the particles, equal to the emissivity of the boundary body, is absorbed and the remaining particles in the packet follow jointly in a newly selected random direction. Upon achievement of a definite level of packet weight (0.001, for example),

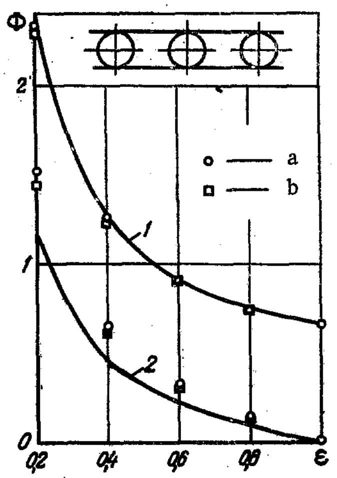

Fig. 2

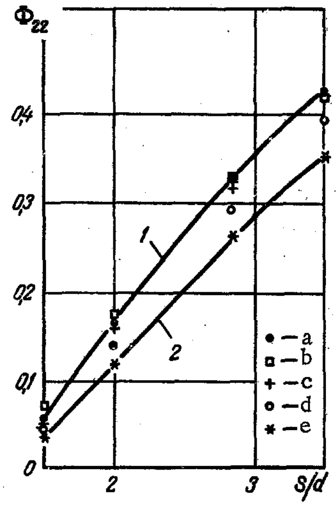

Fig. 3

Fig. 2. Influence of surface emissivity on the mean governing angular coefficients of a tubular system ($s/d = 2$, $e/d = 0.5$, $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon$); curves according to (2): 1) Φ_{21}; 2) Φ_{22}; points according to the Monte Carlo method for diffuse (a) and specular (b) reflection laws (N_2 = 50,000).

Fig. 3. Dependence of the mean governing angular coefficient on the relative tube spacing and the relative distance between the tubes and the masonry; $e/d = 0.5$: 1) exact solution [10], a) Monte Carlo method (N_2 = 50,000), b) the same, specular reflection; $e/d = 0.667$: c) light modeling method [10], d) Monte Carlo method (N_2 = 50,000); $e/d = \infty$: 2) exact solution [10], e) Monte Carlo method (N_2 = 25,000).

tracking it ceases, after which the procedure described is repeated. The ratio between the packet weight N_{ik} which fell on the body k, and the total weight of the packets emitted from the body i and absorbed by the bodies of the system, N_i, is an estimate of the mean governing angular coefficient:

$$\Phi_{ik} = N_{ik}/N_i \quad (i,\ k = 1,\ 2,\ \ldots,\ n). \tag{3}$$

For particles absorbed by the system bodies, the conservation condition which takes the form

$$\sum_{k=1}^{n} \varepsilon_k \Phi_{ik} = 1 \quad (i = 1,\ 2,\ \ldots,\ n) \tag{4}$$

after normalization and agrees with the known closedness condition for the mean governing angular coefficients [7] is satisfied. The coefficients obtained satisfy another compulsory condition — reciprocity — approximately and the experimental nature of the Monte Carlo method is manifest herein in particular.

Let us apply the algorithm elucidated to the investigation of a system (Fig. 1) consisting of a plane series of equidistant infinite parallel tubes or rods 1 set between parallel infinite planes 2 and 3. The radiation in such a system can be assumed planar and the particle motion can be examined within the least element of system symmetry ABCDEF, which has the absolutely specular boundaries AF, BC, and DE. The governing angular coefficients obtained by two approximate methods, algebraic and Monte Carlo, are compared in Fig. 2. It is seen that the Monte Carlo method permits insertion of noticeable corrections to the value of these coefficients, ordinarily used in practice, in a number of cases.

2. Mixed Formulation of the Problem for Diffuse Reflection from a Surface

The temperatures of some number of zones (n_1) and the densities of the resultant radiation in the remaining zones (n_2) are given here. It is recommended in [7] to solve the mixed formulation of the problem by using ones instead of the true values of the reflection coeffi-

TABLE 1. Influence of the Reflection Law on the Governing Angular Coefficient for Coaxial Cylinders with $d_1/d_2 = 0.5$ and $\varepsilon_1 = \varepsilon_2 = 0.5$

Reflection law from a cylinder	Inner	Spec.	Diff.	Spec.	Diff.
	Outer	spec.	spec.	diff.	diff.
Value Coefficient Φ_{12}	Exact	1.333	1.333	1.600	1.600
	Monte Carlo	1.333	1.333	1.599	1.604

cients in the integral equations of the type (1) for the n_2 group of bodies, whereupon formal agreement is achieved between the integral equations to determine the governing angular coefficients in the fundamental and mixed formulations of the problem and, therefore, agreement between the methods of their solution. Hence, the bodies of group n_2 should be considered as totally diffuse reflectors in the Monte Carlo method. The method elucidated above remains unchanged in the remaining algorithm.

The mixed formulation of the problem for the system shown in Fig. 1 occurs, for example, for an absolutely black isothermal plane 2 and tubes 1 and absolutely non-heat-conducting masonry 3 [10]. Its solution is represented in Fig. 3. It is seen that the magnitudes of the coefficients Φ_{22} determined by the Monte Carlo method practically agree with the exact solutions (curves 1 and 2) and that the approximate values of the coefficients obtained by the light modeling and Monte Carlo methods are in good agreement for a relative tube spacing of 1.5 and are somewhat worse for the spacings 2 and 2.833.

3. Fundamental Formulation of the Problem for Specular Reflection from the Surfaces

The appropriate replacement of the reflection law for specular bodies is made in the computational scheme of the Monte Carlo method described in Sec. 1.

As an illustration, let us examine the problem of specular reflection in a coaxial system of two infinite cylinders (Fig. 4). It follows from [8] that the formula

$$\varepsilon_{12} = [\varepsilon_1^{-1} + \varepsilon_2^{-1} - 1]^{-1} \quad (5)$$

is valid for the reduced emissivity of a system in the case of a specularly reflecting outer cylinder and a specularly or diffusely reflecting inner cylinder, and the formula

$$\varepsilon_{12} = [\varepsilon_1^{-1} + d_1/d_2\,(\varepsilon_2^{-1} - 1)]^{-1} \quad (6)$$

is valid for the case of a diffusely reflecting outer and specularly reflecting inner cylinder. The latter expression agrees with the known Christiansen–Nusselt formula for diffusely radiating and reflecting coaxial cylinders. The passage to the governing angular coefficient is made by using the relationship [7]:

$$\varepsilon_{12} = \varepsilon_1\varepsilon_2\Phi_{12}/\varphi_{12}. \quad (7)$$

Presented in Table 1 are values of the coefficient Φ_{12} computed by means of (5)-(7) and determined from an analysis of 15,000 histories. It is seen that the values of the coefficient obtained by the Monte Carlo method are close to its exact values and that the error in the method for a specular reflection law does not exceed its error for a diffuse reflection law.

Points obtained by using the Monte Carlo method under the assumption of specular reflection from all bodies of the system shown in Fig. 2 are superposed there. It is seen that a change in the reflection law results in some change in the magnitude of the governing angular coefficients, up to 8%. The comparatively slight influence of the reflection law on the system characteristics was also noted in [12, 13].

4. Mixed Formulation of the Problem for Specular Reflection from the Surfaces

The solution of this problem by the Monte Carlo method is characterized by a different organization of the reflection process from the specular bodies of the different groups described in Sec. 2. The specular reflection process for the isothermal group of bodies n_1 pro-

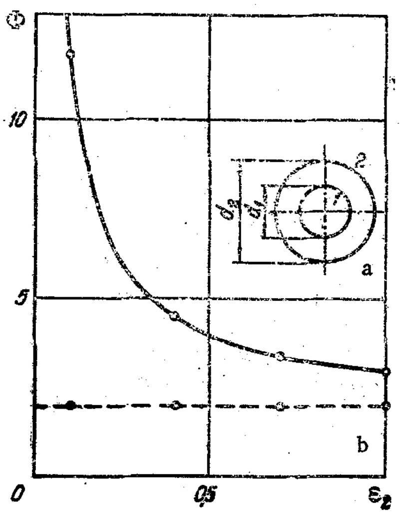

Fig. 4. Coaxial system (a) and influence of the emissivity of a specular surface on the governing angular coefficients of a coaxial system for $d_1/d_2 = 0.5$ and $\varepsilon_1 = 0.5$ (b) [curves are the exact solution according to (8); solid curve is Φ_{22}; dashes are Φ_{21}; points are Monte Carlo method ($N_2 = 5000$)].

ceeds exactly the same as in the preceding formulation of the problem: part of the particles of the approaching packet, equal to the body emissivity, is absorbed by it, the rest follow in the specular reflection direction. The approaching packet is reflected entirely for the specular bodies of the group n_2: diffusely with a probability equal to the body emissivity and specularly with a probability equal to the reflection coefficient of the body.

The algorithm elucidated was verified for the system of coaxial cylinders (Fig. 4). Both cylinders were assumed specular: the inner, isothermal, and the outer, with a given value of the resultant radiation density. This system is described by four governing angular coefficients, for which the expressions can be obtained by using (5), (7), the general solution of the mixed formulation of the problem [7], and also the closedness and reciprocity conditions for these coefficients. We finally have

$$\left.\begin{aligned}&\Phi_{11} = \Phi_{21} = \varepsilon_1^{-1};\ \Phi_{12} = \varepsilon_1^{-1}\, d_2/d_1;\\&\Phi_{22} = [\varepsilon_1^{-1} + \varepsilon_2^{-1}(1 - d_1/d_2) - 1]\, d_2/d_1.\end{aligned}\right\} \tag{8}$$

The solution of the problem is represented in Fig. 4. Practical agreement between the results of the Monte Carlo method and the exact results is observed. It is seen from Fig. 4, as well as from the structure of the algorithm, that the emissivity of a specularly reflecting body on which the resultant radiation density is given influences the governing angular coefficients. Therefore, the emissivity of a locally adiabatic body can act on the thermal characteristics of a system, which is not, as is known, observed for a diffuse reflection law. The influence of the emissivity of an adiabatic wall with specular reflection on its temperature and heat transmission in the system is also noted in [14].

An investigation of the change in the geometric-optics coefficients upon replacement of the diffusely reflecting masonry 3 by specularly reflecting masonry is of practical interest for the configuration shown in Fig. 1. The solution of this problem for the limit case of an adiabatic masonry, reflecting the incident radiation completely, is presented in Fig. 3. It is seen that the passage to specular reflection inserts some corrections to the value of the coefficient, which have a sign dependent on the relative tube spacing. Meanwhile, a change in the radiant heat perception of the tubes turns out to be small in this case — up to 2%.

5. Fundamental and Mixed Formulations of the Problem for Diffuse Radiation and Reflection of a Radiant Flux and a Uniform Reflected Flux Distribution over the Reflecting Body Surface

Such a reflected flux distribution is modeled in the Monte Carlo method by means of an equally probable departure of the reflected part of the packet from any point of the reflecting body surface without a dependence on the point of incidence of this packet on the given body. The method in the rest of the algorithm corresponds to the description given in Secs. 1 and 2. Such a model of the radiant heat-exchange process, extended over all the bodies of the system, corresponds completely to the description of this process by using the approximate system of algebraic equations (2) and permits a strict check on the results obtained by the Monte Carlo method by means of the exact analytic solutions. As is seen from Fig. 3, the points obtained by using the present scheme of the Monte Carlo method practically agree with the curve 2 which is a solution of (2).

On the whole, the results represented in Fig. 3 indicate that the discrepancy between the values of the governing angular coefficient obtained by the Monte Carlo method and the exact solutions does not emerge beyond the maximum value of the error determined, as has been mentioned above, on the basis of the parameters of a normal probability distribution law.

In conclusion, let us note that the algorithms of the Monte Carlo method elucidated for the considered problem formulations (except the last) permit obtaining values of the local governing angular coefficients. To do this, it is sufficient to fix the location of the radiating point on the system surface.

NOTATION

F, surface area; i,j,k, zone numbers; N, number of particles (histories); n, number of zones; $\mathscr{P}$, center of an area element; R, reflection coefficient; s, tube spacing; ε, emissivity; φ, Φ, geometric and geometric-optics governing angular coefficients.

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